The Dirac field in Taub-NUT background

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Abstract

We investigate the $SO(4,1)$ gauge-invariant theory of the Dirac fermions in the external field of the Kaluza-Klein monopole, pointing out that the quantum modes can be recovered from a Klein-Gordon equation analogous to the Schrödinger equation in the Taub-NUT background. Moreover, we show that there is a large collection of observables that can be directly derived from those of the scalar theory. These offer many possibilities of choosing complete sets of commuting operators which determine the quantum modes. In addition there are some spin-like and Dirac-type operators involving the covariantly constant Killing-Yano tensors of the hyper-Kähler Taub-NUT space. The energy eigenspinors of the central modes in spherical coordinates are completely evaluated in explicit, closed form.

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1 Introduction

Hawking [1] has suggested that the Euclidean Taub-NUT metric might give rise to the gravitational analogue of the Yang-Mills instanton. The Euclidean Taub-NUT metric satisfies Einstein's equations with zero cosmological constant. This metric is also involved in many modern studies in physics. For example the Taub-NUT metric is the space part of the line element of the celebrated Kaluza-Klein monopole of Gross and Perry [2] and of Sorkin [3]. On the other hand, in the long-distance limit, neglecting radiation, the relative motion of two monopoles is described by the geodesics of this space [4]. From the mathematical point of view, the Taub-NUT geometry is also very interesting. In the Taub-NUT geometry there are four Killing-Yano tensors [5]. Three of these are complex structure realizing the quaternionic algebra and the Taub-NUT manifold is hyper-Kähler. In addition to these three vector-like Killing-Yano tensors, there is a scalar one and it exists by virtue of the metric being type D.

The Schrödinger quantum modes in the Euclidean Taub-NUT geometry were analyzed using algebraic [6] or analytical methods [7]. The Dirac equation in this background was studied in the mid eighties [8]. It was later realized that the geodesic motion in Euclidean Taub-NUT space is integrable and has a remarkable close similarity with motion under a Coulomb force [5, 6]. For the geodesic motion in the Taub-NUT space, the conserved vector analogous to the Runge-Lenz vector of the Kepler type problem is quadratic in 4-velocities, its components are Stäckel-Killing tensors and they can be expressed as symmetrized product of Killing-Yano tensors.

The fermion problem in the Taub-NUT gravitational instanton field was also studied by Comtet and Horvathy [9] using the observation that the Dirac operator is supersymmetric in 4 dimensions. This property can be connected with fact that the Pauli Hamiltonian for a spin $\frac{1}{2}$ particle in the field of a Dirac magnetic monopole possesses a dynamical supersymmetry [10].

We should like to continue this study in the context of the standard relativistic gauge-invariant theory [11, 12] of the Dirac field in Taub-NUT background [8]. There are some gaps in the previous treatments of the problem which must be filled in. For example, in the Schrödinger case, the existence of the extra conserved quantities of the Runge-Lenz type implies the possibility of separating variables in two different coordinate systems. The presence of the "hidden" symmetry of the Taub-NUT problem must exercise an in-

fluence over the Dirac equation in this background. Finally, we intend to compare the study of the Dirac equation on the Taub-NUT manifold with other treatments of the fermions on curved spaces. For example, there are pseudo-classical models for relativistic spin $\frac{1}{2}$ particle involving anticommuting vectorial degrees of freedom [13]. For the Taub-NUT spinning space, the relations between symmetries, supersymmetries and constants of motion have been investigated in [14]. Spinning particles are in some sense the classical limit of the Dirac particles. The quantization of these models gives rise to supersymmetric quantum mechanics. After quantization, the anticommuting Grassmann variables are mapped into the Dirac matrices while the supercharge becomes just the static part of the Dirac operator (as defined in section 3).

We devote the present paper to the general $SO(4,1)$ gauge-invariant theory of the Dirac fermions [15] in the external field of the Kaluza-Klein monopole[8]. Our goal is to point out new features of this theory and to find the quantum modes determined by complete sets of commuting operators. We start with a gauge-invariant action involving the Dirac field and *pentad* fields (or fünfbein) giving local frames like in Ref.[16]. Thus we preserve the manifest covariance of the theory under space rotations such that the familiar three-dimensional vector notation can be used. Our main result is that the Dirac equation obtained in this way [8] is analytically solvable having quantum modes that can be recovered from those of the Klein-Gordon equation which has the same solutions as the Schrödinger equation [7] but with different parameters. Thus we show that the Dirac equation can be solved in a manner close to that used for deriving the eigenspinors of the static Dirac operator studied in Ref.[9]. Another result is that for each conserved observable of the scalar theory there is at least one operator which commutes with the Dirac Hamiltonian. Consequently, new conserved observables arise including some interesting ones that can be related to the specific objects of the Taub-NUT geometry. As an application, we write down the general form of the energy eigenspinors of the central modes determined by a complete set of commuting operators that includes the Hamiltonian. We obtain a similar energy spectrum as in Ref.[9] but different energy eigenspinors written in terms of new spherical spinors which solve the angular eigenvalue problems.

We start in section 2 with a brief review of the Taub-NUT geometry and the main orbital operators. In the next one we derive the Dirac equation from the standard $SO(4,1)$ gauge-invariant action while in section 4 we show that this theory has well defined properties concerning the supersymmetry and

Hermitian conjugation. In section 5 we solve the Dirac equation while in the next one we discuss the form of the possible conserved observables. Section 7 is devoted to the construction of spin-like and Dirac-type operators in direct connections with the hyper-Kähler structure of the Euclidean Taub-NUT space. In section 8 we give the solutions corresponding to the central modes and, finally, we present our conclusions. In Appendix we introduce the new spherical spinors we need. We work in natural units with $\hbar = c = 1$.

2 Preliminaries

The Kaluza-Klein monopole [2, 3] was obtained by embedding the Taub-NUT gravitational instanton into five-dimensional theory, adding the time coordinate in a trivial way. In a static chart of coordinates x^μ $(\mu, \nu, ... = 0, 1, 2, 3, 5)$, its line element is expressed as

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = dt^2 - \frac{1}{V} dl^2 - V(dx^5 + A_i dx^i)^2 \,. \tag{1}$$

Here $dl^2 = (d\vec{x})^2 = (dx^1)^2 + (dx^2)^2 + (dx^3)^2$ is the usual Euclidean 3-dimensional line element involving the Cartesian physical space coordinates x^i $(i, j, ... = 1, 2, 3)$ which cover the domain D. The other coordinates are the time, $x^0 = t$, and the Cartesian Kaluza-Klein extra-coordinate, $x^5 \in D_5$. The functions V and A_i are static depending only on $\vec{x}$ as

$$\frac{1}{V} = 1 + \frac{\mu}{r}\,, \quad A_1 = -\frac{\mu}{r}\frac{x^2}{r + x^3}\,, \quad A_2 = \frac{\mu}{r}\frac{x^1}{r + x^3}\,, \quad A_3 = 0 \tag{2}$$

where $r = |\vec{x}|$. The regular Taub-NUT metric has the function V with μ positive and it is smooth in the range $r \geq 0$. The case of $\mu < 0$ is also interesting since then scalar modes with discrete energy levels are allowed. In both cases we consider that the space domain of the local chart with Cartesian coordinates is defined by the condition $V > 0$. $\vec{A}$ is the Dirac monopole vector potential giving the magnetic field with central symmetry

$$\vec{B} = \mathrm{rot}\,\vec{A} = \mu \frac{\vec{x}}{r^3} \,. \tag{3}$$

In fact, the spacetime defined by (1) and (2) has the *global* symmetry of the group $G_s = SO(3) \otimes U_5(1) \otimes T_t(1)$ since the line element is invariant under the

global rotations of the Cartesian space coordinates and x^5 and t translations of the Abelian groups $U_5(1)$ and $T_t(1)$ respectively. We note that the $U_5(1)$ symmetry eliminates the so called NUT singularity if x^5 has the period $4\pi\mu$.

The main orbital operators of the relativistic quantum mechanics can be introduced by using the geometric quantization. In this way, one obtains the momentum operators in coordinate representation,

$$P_i = -i(\partial_i - A_i\partial_5)\,, \quad P_5 = -i\partial_5\,. \tag{4}$$

They obey the commutation rules

$$[P_i, P_j] = i\varepsilon_{ijk}B_k P_5\,, \quad [P_i, P_5] = 0\,, \tag{5}$$

and give the Klein-Gordon operator,

$$\nabla_\mu g^{\mu\nu}\nabla_\nu = {\partial_t}^2 + \Delta\,, \quad \Delta = V\vec{P}^{\,2} + \frac{1}{V}{P_5}^2\,, \tag{6}$$

where ∇_μ are the usual covariant derivatives.

Other important orbital operators are the generators of the group G_s defined up to the factor $-i$ as the Killing vector fields corresponding to the global symmetry of the background. Thus the generator of the group $U_t(1)$ is $-i\partial_t$ while the $U_5(1)$ generator is just P_5. The other three Killing vectors give the $SO(3)$ generators which are the components of the orbital angular momentum operator

$$\vec{L} \,=\, \vec{x}\times\vec{P} - \mu\frac{\vec{x}}{r}P_5\,. \tag{7}$$

These generators satisfy the canonical commutation rules among themselves and with the components of all the other vector operators (e.g. coordinates, momenta, etc.).

3 The Dirac Field

Let us denote by $e(x)$ the pentad fields that define the local frames and by $\hat{e}(x)$ those of the corresponding coframes. Their components, which give us the 1-forms $\hat{e}^{\hat{\alpha}} = \hat{e}^{\hat{\alpha}}_\mu dx^\mu$ and the local derivatives $\hat{\partial}_{\hat{\nu}} = e^\mu_{\hat{\nu}}\partial_\mu$, have the usual orthonormalization properties $g_{\alpha\beta}e^\alpha_{\hat{\mu}}\,e^\beta_{\hat{\nu}} = \eta_{\hat{\mu}\hat{\nu}}$, $g^{\alpha\beta}\hat{e}^{\hat{\mu}}_\alpha\,\hat{e}^{\hat{\nu}}_\beta = \eta_{\hat{\mu}\hat{\nu}}$, $\hat{e}^{\hat{\mu}}_\alpha e^\alpha_{\hat{\nu}} = \delta^{\hat{\mu}}_{\hat{\nu}}$, where $\eta = \mathrm{diag}(1,-1,-1,-1,-1)$ is the five-dimensional flat metric. In the

case of the Taub-NUT geometry it is convenient to choose pentad fields with the following non-vanishing components [16]

$$\hat{e}^0_0 = 1\,, \quad \hat{e}^i_j = \frac{1}{\sqrt{V}}\delta_{ij}\,, \quad \hat{e}^5_i = \sqrt{V}A_i\,, \quad \hat{e}^5_5 = \sqrt{V}\,, \tag{8}$$

$$e^0_0 = 1\,, \quad e^i_j = \sqrt{V}\delta_{ij}\,, \quad e^5_i = -\sqrt{V}A_i\,, \quad e^5_5 = \frac{1}{\sqrt{V}}\,. \tag{9}$$

The components of the metric tensor, which can be written as $g_{\mu\nu} = \eta_{\hat{\alpha}\hat{\beta}}\hat{e}^{\hat{\alpha}}_{\mu}\hat{e}^{\hat{\beta}}_{\nu}$ and $g^{\mu\nu} = \eta^{\hat{\alpha}\hat{\beta}}e^{\mu}_{\hat{\alpha}}e^{\nu}_{\hat{\beta}}$, raise or lower only the Greek indices (ranging from 0 to 5) while for the hat Greek ones (with the same range) we have to use the flat metric η. The commutation relations of the derivatives $\hat{\partial}_{\hat{\nu}}$ define the Cartan coefficients $C_{\hat{\mu}\hat{\nu}\cdot}^{\cdot\cdot\hat{\sigma}} = e^{\alpha}_{\hat{\mu}}e^{\beta}_{\hat{\nu}}(\hat{e}^{\hat{\sigma}}_{\alpha,\beta} - \hat{e}^{\hat{\sigma}}_{\beta,\alpha})$, which will help us to write the spin connection in the local frames.

The gauge group of the metric η is $G(\eta) = SO(4,1)$. Its universal covering group, denoted by $\tilde{G}(\eta)$, is a subgroup of the $SU(2,2)$ group which has a fundamental representation just in the space of the four-dimensional Dirac spinors. The five matrices $\tilde{\gamma}^{\hat{\alpha}}$ which must satisfy

$$\left\{\tilde{\gamma}^{\hat{\alpha}},\, \tilde{\gamma}^{\hat{\beta}}\right\} = 2\eta^{\hat{\alpha}\hat{\beta}}, \tag{10}$$

can be defined in terms of standard Dirac matrices [17, 15] as $\tilde{\gamma}^0 = \gamma^0$, $\tilde{\gamma}^i = \gamma^i$, $(i = 1, 2, 3)$, $\tilde{\gamma}^5 = i\epsilon\gamma^5$ where we have introduced the parameter $\epsilon = \pm 1$. This choice has the advantage that all these five matrices are self-adjoint with respect to the usual Dirac conjugation, $\overline{\tilde{\gamma}^{\hat{\alpha}}} = \gamma^0(\tilde{\gamma}^{\hat{\alpha}})^{\dagger}\gamma^0 = \tilde{\gamma}^{\hat{\alpha}}$. Moreover, if we denote by $S^{\hat{\alpha}\hat{\beta}}$ the covariant basis generators of the group $\tilde{G}(\eta)$ then we have

$$\left[\tilde{\gamma}^{\hat{\alpha}},\, \tilde{\gamma}^{\hat{\beta}}\right] = -4iS^{\hat{\alpha}\hat{\beta}}\,, \tag{11}$$

$$\left[S^{\hat{\alpha}\hat{\beta}},\, \tilde{\gamma}^{\hat{\mu}}\right] = i(\eta^{\hat{\beta}\hat{\mu}}\tilde{\gamma}^{\hat{\alpha}} - \eta^{\hat{\alpha}\hat{\mu}}\tilde{\gamma}^{\hat{\beta}})\,. \tag{12}$$

Let ψ be a Dirac free field of mass M, defined on the space domain $D \times D_5$. Its gauge-invariant action [15] has the form

$$\mathcal{S}[\psi] = \int d^5x\sqrt{g}\left\{\frac{i}{2}\left[\overline{\psi}\tilde{\gamma}^{\hat{\mu}}\tilde{\nabla}_{\hat{\mu}}\psi - (\overline{\tilde{\nabla}_{\hat{\mu}}\psi})\tilde{\gamma}^{\hat{\mu}}\psi\right] - M\overline{\psi}\psi\right\} \tag{13}$$

involving the spin covariant derivatives $\tilde{\nabla}_{\hat{\mu}} = \hat{\partial}_{\hat{\mu}} + \Gamma_{\hat{\mu}}$ whose spin connection matrices are

$$\Gamma_{\hat{\mu}} = \frac{i}{4} S^{\hat{\nu}\hat{\lambda}} (C_{\hat{\mu}\hat{\nu}\hat{\lambda}} + C_{\hat{\lambda}\hat{\mu}\hat{\nu}} + C_{\hat{\lambda}\hat{\nu}\hat{\mu}}) \,. \tag{14}$$

Starting with Eqs.(8) and (9), after a little calculation, we find

$$\tilde{\nabla}_i = i\sqrt{V} P_i + \frac{i}{2} V\sqrt{V} \varepsilon_{ijk} \Sigma^*_j B_k \,, \quad \tilde{\nabla}_5 = \frac{i}{\sqrt{V}} P_5 - \frac{i}{2} V\sqrt{V} \vec{\Sigma}^* \cdot \vec{B} \,, \tag{15}$$

where

$$\Sigma^*_i = S_i + \frac{\epsilon}{2} \gamma^5 \gamma^i \,, \quad S_i = \frac{1}{2} \varepsilon_{ijk} S^{jk} \,. \tag{16}$$

Finally we obtain the Dirac equation, $\mathcal{D}\psi = 0$, given by the Dirac operator [8]

$$\begin{aligned} \mathcal{D} &= i\tilde{\gamma}^{\hat{\alpha}} \tilde{\nabla}_{\hat{\alpha}} - M = i\gamma^0 \partial_t - \mathcal{D}_s \\ &= i\gamma^0 \partial_t - \sqrt{V} \vec{\gamma} \cdot \vec{P} - \frac{i\epsilon}{\sqrt{V}} \gamma^5 P_5 - \frac{i\epsilon}{2} V\sqrt{V} \gamma^5 \vec{\Sigma}^* \cdot \vec{B} - M \,, \end{aligned} \tag{17}$$

which has the usual time-dependent term [17] and a *static* part, $\mathcal{D}_s$, determined by the Taub-NUT geometry. The parameter values $\epsilon = \pm 1$ define two distinctive versions of the theory related between themselves through

$$\mathcal{D}(-\epsilon, M) = -\gamma^5 \mathcal{D}(\epsilon, -M) \gamma^5 \,. \tag{18}$$

Hereby it results that these can be considered as chiral transformations to each other *only* when $M = 0$. We observe that in both cases the matrices Σ^*_i are singular such that $\Sigma^*_i(\epsilon) + \Sigma^*_i(-\epsilon) = 2S_i$. This is the motive why these versions appear as having quite different formalisms even if, in fact, they are equivalent as it results from Eq.(18). In order to do not produce confusion, in the following we work only in the version with $\epsilon = 1$. We note that the Dirac operator of Ref.[9] is the static operator $\mathcal{D}_s$ of the version with $\epsilon = -1$ (and $M = 0$).

By definition [17], the Hamiltonian operator is $H = \gamma^0 \mathcal{D}_s$. Other important observables are the generators of the global symmetry, P_5 and the whole angular momentum operator $\vec{J} = \vec{L} + \vec{S}$. One can verify that its components, J_i, as well as P_5 are conserved in the sense that they commute with $\mathcal{D}$ and H. This means that, as was expected, the Dirac equation is covariant under the transformations of the group G_s.

4 The regular Dirac equation

The Dirac equation can be studied in the original above representation (with fixed $\epsilon = 1$) or in any other representation obtained through a transformation, $\psi \to \psi' = T\psi$, which can be unitary or not. Thus one can find suitable representations where the behavior of the main operators under Hermitian conjugation or the supersymmetry of the Hamiltonian operator should be better pointed out.

In our case it is convenient to write the Dirac equation in *regular* form with the help of the transformation $\psi_R = V^{-1/4}\psi$ [8]. Using the identity [9]

$$V^{\alpha} P_i V^{-\alpha} = P_i + i\alpha V B_i \,, \tag{19}$$

we obtain the regular Dirac equation $\mathcal{D}_R \psi_R = 0$ given by the operator

$$\mathcal{D}_R = i\gamma^0\partial_t - \sqrt{V}\vec{\gamma}\cdot\vec{P} - \frac{i}{\sqrt{V}}\gamma^5 P_5 - \frac{i}{2} V\sqrt{V}\gamma^5 \vec{S}\cdot\vec{B} - M \,. \tag{20}$$

In this form the Dirac equation has usual terms apart the specific Kaluza-Klein one and the magnetic term which couples the spin with the magnetic field like in the Schrödinger-Pauli non-relativistic theory.

From the general theory of the Dirac field [12, 15] we can deduce that the time-independent relativistic scalar product of two regular Dirac spinors is

$$(\psi_R, \psi'_R) = \int_D \frac{d^3x}{\sqrt{V}} \int_{D_5} dx^5\, \overline{\psi}_R(x)\gamma^0 \psi'_R(x) \,. \tag{21}$$

Now we see that the regular form has the advantage of manifestly pointing out that the operators $\sqrt{V}P_i$ are self-adjoint with respect to the scalar product (21). Moreover, it is not difficult to convince ourselves that J_i and P_5 are also self-adjoint operators.

The regular Hamiltonian operator, H_R, is given by $\mathcal{D}_R = \gamma^0(i\partial_t - H_R)$. With the standard form of the Dirac matrices [17],

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}, \quad \gamma^5 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{22}$$

(where σ_i are the Pauli matrices), we obtain

$$H_R = \begin{pmatrix} M & \Pi^{\dagger} \\ \Pi & -M \end{pmatrix} \tag{23}$$

where

$$\begin{aligned}\Pi &= \sqrt{V}\vec{\sigma}\cdot\vec{P} - \frac{i}{\sqrt{V}}P_5 - \frac{i}{4}V\sqrt{V}\vec{\sigma}\cdot\vec{B} \\ &= V^{-1/4}\left(\sqrt{V}\vec{\sigma}\cdot\vec{P} - \frac{i}{\sqrt{V}}P_5\right)V^{1/4}\,, \end{aligned} \tag{24}$$

as it results from Eq.(19).

In this representation it is obvious that H_R is self-adjoint and has *manifest* supersymmetry. This means that the two-component spinors of $\psi_R = (\psi_R^{(1)},\, \psi_R^{(2)})^T$ satisfy the equations with second order derivatives,

$$(\Pi^\dagger\,\Pi + M^2 + {\partial_t}^2)\psi_R^{(1)} = 0\,, \quad (\Pi\,\Pi^\dagger + M^2 + {\partial_t}^2)\psi_R^{(2)} = 0\,, \tag{25}$$

that can be obtained directly from $({H_R}^2 + {\partial_t}^2)\psi_R = 0$.

In the original form of the Dirac equation these properties appear as partially hidden since the transformation which leads to the regular equation is not unitary. Other interesting equivalent forms of the Dirac equation can be obtained by using also non-unitary transformations. For example the transformation $T = \mathrm{diag}(1,\, V^{-1/2})$ leads to a representation in which the magnetic term is completely absorbed. However, what is important is that there is at least one representation, namely the regular one, where we find good Hermitian properties and the manifest supersymmetry of the Hamiltonian.

5 The energy eigenspinors

Bearing in mind that the energy, E, is conserved it is convenient to consider particular solutions with fixed energy that can be of positive or negative frequency like in the case of the Dirac theory in flat spacetime. We denote them by

$$\psi_E^{(+)}(x) = e^{-iEt}u_E(\vec{x}, x^5)\,, \quad \psi_E^{(-)}(x) = e^{iEt}v_E(\vec{x}, x^5)\,, \tag{26}$$

and say that u_E is the particle energy eigenspinor while v_E is the antiparticle one since they satisfy $(H-E)u_E = 0$ and $(H+E)v_E = 0$.

Let us analyze the particle-like eigenvalue problems starting with the Hamiltonian operator put in the form

$$H = \left(\begin{array}{cc} M & V\pi^*\frac{1}{\sqrt{V}} \\ \sqrt{V}\pi & -M \end{array}\right) \tag{27}$$

where we have denoted by

$$\pi = \vec{\sigma} \cdot \vec{P} - \frac{iP_5}{V}\,, \quad \pi^* = \vec{\sigma} \cdot \vec{P} + \frac{iP_5}{V}\,, \tag{28}$$

the operators which give

$$\Delta = V\,\pi^*\pi\,. \tag{29}$$

Then we find that the two-component spinors of $u_E = (u_E^{(1)}, u_E^{(2)})^T$ satisfy the following equations

$$V\pi^* \frac{1}{\sqrt{V}}\,u^{(2)} = (E - M)\,u^{(1)}\,, \tag{30}$$

$$\sqrt{V}\pi\,u^{(1)} = (E + M)\,u^{(2)}\,, \tag{31}$$

which, according to Eq.(29), are equivalent with

$$\Delta\,u_E^{(1)} = (E^2 - M^2)\,u_E^{(1)}\,, \tag{32}$$

$$u_E^{(2)} = \frac{1}{E+M}\sqrt{V}\,\pi\,u_E^{(1)}\,. \tag{33}$$

This remarkable result shows us that $u_E^{(1)}$ may be *any* solution of the scalar equation (32) which is nothing else than the static Klein-Gordon equation in Taub-NUT geometry. In other respects, it is clear that $u_E^{(2)}$ is completely determined by Eq.(33) if the form of $u_E^{(1)}$ is given.

Hence the conclusion is that the Dirac equation in Taub-NUT geometry is always analytically solvable since the Klein-Gordon equation (32) is solvable having similar solutions as the Schrödinger one [7]. The selection of the solutions can be done starting with desired spinors $u_E^{(1)}$ that satisfy Eq.(32) and then calculating $u_E^{(2)}$. Since the Klein-Gordon equation is scalar, the form of the spinor $u_E^{(1)}$ and, therefore, that of u_E is strongly dependent on the choice of the spin observables included in the complete set of the commuting operators which defines the quantum modes.

6 Observables

The knowledge of the general form of the solutions offers us the opportunity to analyze the structure and properties of the conserved observables. These are operators in the spaces of the spinors u_E which can be

expressed in terms of operators acting upon the two-component spinors [10], π, π^*, ${}_2J_i = L_i + \sigma_i/2$ and

$$\sigma_r = \frac{1}{r}\vec{\sigma}\cdot\vec{x}\,, \quad \sigma_P = \vec{\sigma}\cdot\vec{P}\,, \quad \sigma_L = \vec{\sigma}\cdot\vec{L}\,. \tag{34}$$

The problem we briefly investigate here is how may look the operators which commute with the Hamiltonian (27).

The simplest operators have the diagonal form

$$X = \left(\begin{array}{cc} X^{(1)} & 0 \\ 0 & X^{(2)} \end{array}\right) \tag{35}$$

where $X^{(1)}$ and $X^{(2)}$ are 2×2 matrix operators. The condition $[X,\, H] = 0$ is equivalent with

$$X^{(2)}\sqrt{V}\pi = \sqrt{V}\pi X^{(1)}\,, \quad V\pi^*\frac{1}{\sqrt{V}}X^{(2)} = X^{(1)}V\pi^*\frac{1}{\sqrt{V}}\,, \tag{36}$$

from which it results that $[X^{(1)},\, \Delta] = 0$. Therefore, we can start with an operator $X^{(1)}$ which commutes with Δ and to try to find the operator $X^{(2)}$ which satisfies Eqs.(36). If this problem has solution and we have

$$X^{(1)}u_E^{(1)} = \xi\, u_E^{(1)}\,, \tag{37}$$

then the energy eigenspinor u_E is an eigenspinor of X corresponding to the same eigenvalue, ξ. The best example is represented by the components of the whole angular momentum, $J_i = \mathrm{diag}({}_2J_i,\, {}_2J_i)$, which commute with H since ${}_2J_i$ commute with π, π^* and V.

If we have an operator $X^{(1)}$ which commutes with Δ and satisfies Eq.(37) but there is no satisfactory solution for $X^{(2)}$ we can introduce an observable starting with the singular operator $X_0 = \mathrm{diag}(X^{(1)},\, 0)$ that commutes with H^2. This property guarantees that the new operator

$$\mathcal{Q}(X^{(1)}) = \{X_0,\, H\}\;, \tag{38}$$

commutes with H. In addition, by using Eqs.(30), (31) and (37) we obtain the interesting eigenvalue equation

$$\mathcal{Q}(X^{(1)})u_E = (E+M)\,\xi\, u_E\,. \tag{39}$$

Thus we arrive to another important result namely that for any operator which commutes with Δ there exists an operator in the space of the Dirac energy eigenspinors u_E of the form (38) which commutes with H. Particularly, for $X^{(1)} = 1$ we get $\mathcal{Q}(1) = H + M$. On the other hand, the mapping $X^{(1)} \to \mathcal{Q}(X^{(1)})$ cannot be interpreted as a representation since, in general, for two operators $X^{(1)}$ and $Y^{(1)}$ which commute with Δ, we have $[\mathcal{Q}(X^{(1)}),\, \mathcal{Q}(Y^{(1)})] \neq \mathcal{Q}([X^{(1)},\, Y^{(1)}])$. Fortunately, if $X^{(1)}$ and $Y^{(1)}$ commute with each other then it is true that $[\mathcal{Q}(X^{(1)}),\, \mathcal{Q}(Y^{(1)})] = 0$.

However, apart from the above cases, there are many other types of conserved operators constructed by using algebraic operations or directly related to the specific geometric objects of the Taub-NUT geometry, as Stäckel-Killing and Killing-Yano tensors. In the next Section we shall present some operators connected with the covariantly constant Killing-Yano tensors of the Taub-NUT geometry.

7 Spin-like and Dirac-type operators

As observed in [5], the Taub-NUT geometry possesses four Killing-Yano tensor of valence 2:

$$f_{\mu(\nu;\lambda)} = 0 \tag{40}$$

where the round bracket denotes symmetrization over the indices enclosed. The first three,

$$\begin{aligned} f^i &= f^i_{\hat\alpha\hat\beta}\hat e^{\hat\alpha} \wedge \hat e^{\hat\beta} \\ &= 2\mu(dx^5 + \vec{A}\cdot d\vec{x}) \wedge dx^i + \varepsilon_{ijk}V^{-1}dx^j \wedge dx^k \\ &= 2\hat e^5 \wedge \hat e^i + \varepsilon_{ijk}\hat e^j \wedge \hat e^k\,, \end{aligned} \tag{41}$$

are rather special since they are covariantly constant (with vanishing field strength).

Using these tensors we can construct the following spin-like operators,

$$\Sigma_i = \frac{i}{4} f^i_{\hat\alpha\hat\beta}\gamma^{\hat\alpha}\gamma^{\hat\beta} = S_i - \frac{1}{2}\gamma^5\gamma^i = \begin{pmatrix} \sigma_i & 0 \\ 0 & 0 \end{pmatrix}, \tag{42}$$

which have similar properties to that of the Pauli matrices σ_i and, in addition, satisfy

$$\Sigma_i + \Sigma_i^* = 2S_i\,. \tag{43}$$

Since the Pauli matrices commute with Δ, the spin-like operators (42) commute with H^2 and, therefore, we can introduce the operators of the form (38)

$$\mathcal{Q}(\sigma_i) = \{\Sigma_i,\, H\} = \left(\begin{array}{cc} 2M\sigma_i & V\sigma_i\pi^*\frac{1}{\sqrt{V}} \\ \sqrt{V}\pi\sigma_i & 0 \end{array}\right) \tag{44}$$

which commute with the Hamiltonian (27). Let us observe that they have the remarkable property:

$$\{\mathcal{Q}(\sigma_i),\, \mathcal{Q}(\sigma_j)\} = 2\delta_{ij}(H+M)^2\,. \tag{45}$$

The existence of the Killing-Yano tensors (41) allows one to construct generalizations of the operator $\mathcal{D}_s$ defined by

$$\mathcal{D}_i = f^i_{\hat\alpha\hat\beta}\tilde\gamma^{\hat\alpha}\tilde\nabla^{\hat\beta}\,, \tag{46}$$

and called Dirac-type operators [14]. After a little calculation we find that their explicit form is

$$\mathcal{D}_i = \left(\begin{array}{cc} 0 & V\sigma_i\pi^*\frac{1}{\sqrt{V}} \\ \sqrt{V}\pi\sigma_i & 0 \end{array}\right) \tag{47}$$

and from Eqs.(42), (44) and (47) we get

$$\mathcal{Q}(\sigma_i) = 2M\Sigma_i + \mathcal{D}_i\,. \tag{48}$$

In addition, we have

$$\{\gamma^0,\, \mathcal{D}_i\} = 0\,, \tag{49}$$

and

$$[H,\, \Sigma^i] = -\gamma^0\mathcal{D}_i\,, \quad [H,\, \mathcal{D}_i] = 2M\gamma^0\mathcal{D}_i\,. \tag{50}$$

For $M=0$, the operators $\mathcal{Q}(\sigma_i)$ coincide with the Dirac-type operators $\mathcal{D}_i$. Moreover, including into Eq.(45) the static part of $\mathcal{D}$, denoted now by $Q_0 = i\mathcal{D}_s = i\gamma^0 H$, by the side of the operators $Q_i = \mathcal{Q}(\sigma_i)$ we obtain

$$\{Q_A,\, Q_B\} = 2\delta_{AB}H^2\,, \quad A, B, ... = 0, 1, 2, 3\,. \tag{51}$$

It is worthy of notice that these relations correspond to the $N=4$ supersymmetry algebra realized by the supercharges of the spinning Taub-NUT model

[5, 6, 14]. In fact relations of the form (51) make manifest the link between the existence of the covariantly constant Killing-Yano tensors (41) and the hyper-Kähler geometry of the Taub-NUT manifold. In other respects, let us observe that Eq.(49) permits to convert the anticommutator (51) into a commutation relation between the operators $\gamma^0 Q_A$ and Q_B for $A \neq B$.

The fourth Killing-Yano tensor of the Taub-NUT space is not covariantly constant. This tensor is involved in the construction of a conserved vector analogous to the Runge-Lenz vector of the Kepler problem [5, 6, 14]. The existence of the extra conserved quantities, quadratic in velocities, implies the possibility of separating variables for Dirac equation in two different coordinate systems. This matter will be discussed elsewhere [18].

8 Central modes

In this section we shall show in what manner the central modes can be defined using an appropriate complete set of commuting observables. We consider the local chart with spherical coordinates, r, θ, ϕ, commonly related to the Cartesian ones and the new coordinate χ defined as

$$\chi = -\frac{1}{\mu}x^5 - \arctan\frac{x^2}{x^1}\,. \tag{52}$$

Then the line element reads

$$ds^2 = dt^2 - \frac{1}{V}(dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta\, d\phi^2) - \mu^2 V(d\chi + \cos\theta\, d\phi)^2\,, \tag{53}$$

since $A_r = A_\theta = 0$ and $A_\phi = \mu(1 - \cos\theta)$. Note that in this chart $r \in D_r = \{r|V(r) > 0\}$ (i.e., $r > 0$ if $\mu > 0$ or $r > |\mu|$ if $\mu < 0$), the angular coordinates θ, ϕ cover the sphere S^2 while $\chi \in D_\chi = [0, 4\pi)$.

In the following it is convenient to use the orbital operators in spherical coordinates [7] and $Q = -\mu P_5 = -i\partial_\chi$ instead of P_5. We assume that the central modes of the Dirac field in Taub-NUT geometry are given by the common eigenspinors of the *complete* set of commuting operators $\{H,\, Q,\, \vec{J}^2,\, J_3,\, \mathcal{Q}(\sigma_L + 1)\}$. These eigenspinors have the form

$$u^{\pm}_{E,q,j,m_j} = N^{\pm}_{E,q,j}\frac{1}{r}\left(\begin{array}{c} -i(E+M)\, f^{\pm}_{E,q,j}\,\Psi^{\pm}_{q,j,m_j} \\ \sqrt{V}(h^{\pm}_{E,q,j}\,\Psi^{\pm}_{q,j,m_j} + g^{\pm}_{E,q,j}\,\Psi^{\mp}_{q,j,m_j}\,) \end{array}\right), \tag{54}$$

where $N^{\pm}$ are normalization factors, the radial functions $f^{\pm}$, $g^{\pm}$ and $h^{\pm}$ depend only on r while $\Psi^{\pm}$ are the spherical spinors defined in Appendix. These completely solve the common eigenvalue problems of the operators Q, $\vec{J}^2$, J_3 and $\mathcal{Q}(\sigma_L + 1)$ for the eigenvalues q, $j(j+1)$, m_j, and $\pm(E + M)(j + \frac{1}{2})$, respectively.

Then it remains only to solve the energy eigenvalue problem which must give the radial functions according to Eqs.(32) and (33). We find first that $f^{\pm}$ is a solution of the radial Klein-Gordon equation,

$$\left[-\frac{d^2}{dr^2} + \frac{l_{\mp}(l_{\mp}+1)}{r^2} - \frac{\alpha}{r}\right] f^{\pm}_{E,q,j}(r) = \beta f^{\pm}_{E,q,j}(r)\,, \tag{55}$$

which has the *same* form as the Schrödinger one [7] but with the parameters, $l_{\pm} = j \pm \frac{1}{2}$ and

$$\alpha = \mu\left[E^2 - M^2 - 2\frac{q^2}{\mu^2}\right]\,, \quad \beta = E^2 - M^2 - \frac{q^2}{\mu^2}\,. \tag{56}$$

The other radial functions can be calculated from Eq.(33) if we take into account that

$$i\sigma_P = \sigma_r\left[\partial_r + \frac{1}{r} - \frac{\sigma_L + 1}{r}\right] + \frac{Q}{r} \tag{57}$$

and by using Eqs.(67) and (71). We obtain the radial equations

$$g^{\pm}_{E,q,j} = \sqrt{1-(\lambda^q_j)^2}\left(-\frac{d}{dr} \pm \frac{j+\frac{1}{2}}{r}\right) f^{\pm}_{E,q,j}\,, \tag{58}$$

$$h^{\pm}_{E,q,j} = \lambda^q_j\left(\mp\frac{d}{dr} + \frac{j+\frac{1}{2}}{\mu V}\right) f^{\pm}_{E,q,j}\,, \tag{59}$$

which lead to the identity

$$h^{\pm}_{E,q,j} = \frac{q}{\mu}\, f^{\pm}_{E,q,j} \pm \frac{\lambda^q_j}{\sqrt{1-(\lambda^q_j)^2}}\, g^{\pm}_{E,q,j}\,. \tag{60}$$

The last step is to calculate the normalization constants with the help of the scalar product (21) where the spinors ψ_R will be replaced by $V^{-1/4}u^{\pm}_{E,q,j,m_j}$. Since the spherical spinors are orthogonal, we obtain for the eigenspinors corresponding to the discrete energy levels that

$$\frac{1}{|N^{\pm}_{E,q,j}|^2} = (E+M)^2\int_{D_r}\frac{dr}{V}|f^{\pm}_{E,q,j}|^2 + \int_{D_r} dr\,\left(|g^{\pm}_{E,q,j}|^2 + |h^{\pm}_{E,q,j}|^2\right)\,. \tag{61}$$

The radial Klein-Gordon equation (55) produces the same central modes as those discussed in Ref.[7] with similar energy spectra. Therefore the energy levels are deeply degenerated having no fine structure. For example, the discrete levels of the case $\mu < 0$,

$$E_n^2 = M^2 + \frac{2}{\mu^2}\left[n\sqrt{n^2-q^2} - (n^2-q^2)\right] , \tag{62}$$

depend only on the principal quantum number n which takes all the integer values allowed by the selection rule $|q| < j + \frac{1}{2} < n$ derived from those presented in Appendix. It is interesting to observe that the condition $n > |q|$ eliminates the zero modes (with $E = M$) for $q \neq 0$. Like in the scalar case the energy spectrum is countable with finite range since

$$\lim_{n\to\infty} E_n^2 = {M_{ef}}^2 = M^2 + \frac{q^2}{\mu^2} , \tag{63}$$

where M_{ef} play the role of the effective mass of the spin half particle in the field of the Dirac monopole. This arises from the usual mass term of $\mathcal{D}$ combined with the specific Kaluza-Klein contribution [15, 8]. We must specify that in the case of $M = 0$ this spectrum is similar with that of Ref.[9] but the energy eigenspinors as well as the selection rules of the quantum numbers are different.

Finally we observe that there are more other conserved observables which commute with those of the complete set that defines the central modes. Indeed, if we take the operators $\vec{L}^2$, ${}_2\vec{J}^2$ and ${}_2J_3$ which commute with Δ, we can verify that the operators $\mathcal{Q}(\vec{L}^2)$, $\mathcal{Q}({}_2\vec{J}^2)$ and $\mathcal{Q}({}_2J_3)$ commute among themselves and with all the other observables considered above. Consequently, the eigenspinors (54) of the central basis are, in addition, common eigenspinors of these new observables, corresponding to the eigenvalues $(E+M)l_{\mp}(l_{\mp}+1)$, $(E+M)j(j+1)$ and $(E+M)m_j$, respectively.

9 Conclusions

The main conclusion of this article is that the $SO(4,1)$ gauge-invariant theory of the Dirac field in Taub-NUT geometry leads to an analytically solvable field equation which gives similar energy levels like those of the scalar modes. In general, these are deeply degenerated having no fine structure. For

this reason we need to use many commuting operators in order to completely determine the quantum modes of the Dirac field. Fortunately, in this theory we have to handle a collection of conserved observables much more larger than that of the scalar field. This is because one can associate to each operator which commutes with the Klein-Gordon operator at least one operator which should commute with the Dirac Hamiltonian. In this way we have found the new operator $\mathcal{Q}(\sigma_L+1)$ necessary to complete the set of commuting operators $\{H,\, Q,\, \vec{J}^2,\, J_3\}$ and the other new conserved observables that are diagonal in the basis of the eigenspinors (54). However, as mentioned, there is the possibility that other kind of conserved observables could be constructed using the specific Killing-Yano tensors of the Taub-NUT geometry.

In a forthcoming paper [18] we shall analyze the consequences of the presence of a conserved vector analogous to the Runge-Lenz vector of the Kepler-type problem. The existence of this conserved quantity is rather surprising in view of the complexity of the equations of geodesic motion in the Taub-NUT space. The components of the Runge-Lenz type vector are Killing tensors and they can be expressed as symmetrized products of the covariantly constant Killing-Yano tensors f^i (41) and a fourth Killing-Yano tensor f^Y non-covariantly constant [5, 14]. The first three tensors, f^i, transform as vector under spatial rotations generated by $\vec{J}$, while the last one, f^Y, is a scalar. The existence of the extra conserved quantities, quadratic in four-velocities, implies the possibility of separating variables in two different coordinate systems.

Finally we intend to extend the study of the Dirac equations on generalized Taub-NUT metrics [19]. This class of metrics admits a Kepler-type symmetry, but, in general, the Killing tensors involved in the Runge-Lenz vector cannot be expressed as products of Killing-Yano tensors [20]. It is expected that the non-existence of the Killing-Yano tensors makes the study of "hidden" symmetries more laborious for relativistic particles with spin $\frac{1}{2}$.

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A Spherical spinors

We define the spherical spinors in usual manner [17] starting with our $SO(3)\otimes U(1)$ harmonics, $Y^q_{l,m}$, introduced in Ref.[7]. These are more general than others [21] since they are defined for any real value of q without restrictions upon the values of l and m apart the selection rule $|q|-1<|m|\le l$.

In the case of the Taub-NUT geometry the boundary conditions on $S^2\times D_\chi$ require l and m to be integer numbers while $q=0,\pm1/2,\pm1,...$ [6]. However, for integer l the spherical spinors can be defined for any q according to the usual method of the decomposition of the direct product of $SU(2)$ representations by taking the $SO(3)\otimes U(1)$ harmonics instead of the usual ones. Thus we find that the spherical spinors, $\Psi^{\pm}_{q,j,m_j}(\theta,\phi,\chi)$, are the common two-component eigenspinors of the eigenvalue problems

$$\begin{aligned} Q\,\Psi^{\pm}_{q,j,m_j} &= q\,\Psi^{\pm}_{q,j,m_j}\,, && (64)\\ {}_2\vec{J}^2\,\Psi^{\pm}_{q,j,m_j} &= j(j+1)\,\Psi^{\pm}_{q,j,m_j}\,, && (65)\\ {}_2J_3\,\Psi^{\pm}_{q,j,m_j} &= m_j\,\Psi^{\pm}_{q,j,m_j}\,, && (66)\\ (\sigma_L+1)\,\Psi^{\pm}_{q,j,m_j} &= \pm(j+1/2)\,\Psi^{\pm}_{q,j,m_j}\,. && (67)\end{aligned}$$

They are, in addition, eigenfunctions of $\vec{L}^2$ corresponding to the eigenvalues $l(l+1)$ with $l=j\pm\frac{1}{2}$. For $j=l+\frac{1}{2}>|q|-\frac{1}{2}$ we have [17, 7]

$$\Psi^{+}_{q,j,m_j}=\frac{1}{\sqrt{2j}}\left(\begin{array}{c}\sqrt{j+m_j}\,Y^q_{j-\frac{1}{2},m_j-\frac{1}{2}}\\ \sqrt{j-m_j}\,Y^q_{j-\frac{1}{2},m_j+\frac{1}{2}}\end{array}\right) \tag{68}$$

while for $j=l-\frac{1}{2}>|q|-\frac{3}{2}$ we get

$$\Psi^{-}_{q,j,m_j}=\frac{1}{\sqrt{2j+2}}\left(\begin{array}{c}\sqrt{j-m_j+1}\,Y^q_{j+\frac{1}{2},m_j-\frac{1}{2}}\\ -\sqrt{j+m_j+1}\,Y^q_{j+\frac{1}{2},m_j+\frac{1}{2}}\end{array}\right)\,. \tag{69}$$

We specify that these spherical spinors are orthonormal since the $SO(3)\otimes U(1)$ harmonics are orthonormal with respect to the angular scalar product [7].

Finally, using Eq.(67) and the superalgebra

$${\sigma_r}^2=1\,,\quad \{\sigma_r,\,\sigma_L+1\}=2q \tag{70}$$

we find that

$$\sigma_r \Psi^{\pm}_{q,j,m_j} = \pm\lambda^q_j \, \Psi^{\pm}_{q,j,m_j} + \sqrt{1-(\lambda^q_j)^2}\, \Psi^{\mp}_{q,j,m_j} \tag{71}$$

where we have denoted $\lambda^q_j = q/(j+\frac{1}{2})$. Note that a similar property is reported in Ref.[22].

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